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The Uniqueness of g_{ij} in Terms of R^l_{ijk}

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Abstract

Some theorems are given which show when a curvature tensor is the curvature of pseudo-metric and when this pseudo-metric is unique. These results not only contribute to the exact solution work done previously but also throw some light on Mach's principle in general relativity.

1. *Introduction*

The question of when solutions to Einstein's equations exist and whether they are unique if they do exist is of fundamental interest in general relativity. The first approach to this problem has been to translate Einstein's equations into a Cauchy problem in differential equations. This method cannot answer the question of when a given tensor is the stress-energy tensor of some space-time.

Recently Schmidt (1973) has taken a more geometrical approach to the problem and has arrived at a first step in its solution. He found a necessary and sufficient condition for a connection Γ^k_{ij} to be the connection of a Lorentz metric. In the case when this condition is satisfied he gives an explicit way of constructing the metric from the connection. Thus he solves the existence problem completely. The only unfortunate aspect of his construction is that it still necessitates the solution of differential equations, namely the equations of parallel transport in the connection. He also solves the uniqueness problem by finding a stronger condition which is necessary and sufficient for the Lorentz metric to be unique to within a constant multiple. Thus, he has completely solved the problem of existence and uniqueness as far as the gravitational forces Γ^k_{ij} is concerned.

We have made a further step towards an existence and uniqueness theorem concerning stress-energy tensor by dealing with the gravitational fields R^l_{ijk} (Ihrig, 1975a). We showed that under a certain condition on the R^l_{ijk} the g_{ij} are uniquely determined to within a conformal factor by the R^l_{ijk}. Moreover, one can also actually construct an explicit general formula giving the g_{ij} in terms of the R_{ijk}^{l}. This formula involves only algebraic combinations of the R_{ijk}^{l}, and no differential equations need be solved as with Schmidt's procedure. Thus, given the R_{ijk}^{l}, the only unknown factor left is the conformal factor.

Here we would like to consider the uniqueness of the conformal factor. We find in Section 2 that the conformal factor is also unique as long as the Riemann tensor satisfies a slightly stronger condition than needed before. This result has two disadvantages, however. One is that the uniqueness theorem is not constructive. This means one cannot write down the conformal factor explicitly as a function of the R_{ijk}^{l}, but the situation is simplified in that one can find the conformal factor as a solution to a first order linear differential equation. Thus, this theorem together with Ihrig (1975a) seems to provide fairly well for the complete solution of the g_{ij} in terms of the R_{ijk}^{l} anyway. The second disadvantage is that the extra condition needed for the uniqueness does not have a clear geometric interpretation. This is not much of a problem in terms of actual computation since the condition holds for almost every space-time (the condition is generic) but the situation is unsatisfactory in so far as one is not certain what kind of physical assumption the condition reflects. Thus in Section 3 we give another uniqueness theorem which involves the holonomy group instead of R_{ijk}^{l} but does not use this extra assumption. It is this theorem that shows what geometry is involved in determining the conformal factor and serves to show why the technique used in Section 2 would be expected to work.

Besides the uniqueness theorems in Section 2 and Section 3 we find that because the uniqueness theorems of Section 2 and Ihrig (1975a) are so explicit we can give an easy existence theorem. The theorem gives a necessary and sufficient condition for a given tensor R_{ijk}^{l} to be the curvature of some Lorentz metric as long as the given R_{ijk}^{l} satisfies two generic conditions. Since some notation will be needed from Ihrig (1975a) in this theorem, this notation together with a key result is given in the Appendix. This will serve to make this article relatively self-contained.

In Section 4 we would like to take some time to discuss the physical significance of these results. It is perhaps fairly clear that the results are of interest to physicists from the point of view of computing exact solutions and to mathematicians from the point of view of finding when tensors are curvatures of a connection (these results hold for Riemannian metrics as well as Lorentz metrics). What is perhaps not so clear is that these theorems play a role in the conceptual foundations of general relativity. They are in fact related to the problem of making Mach's principle compatible with relativity. We discuss this problem in Section 4.

2. *The Conformal Factor in Terms of* R_{ijk}^{l}

First we give a theorem that shows when the conformal factor is determined in terms of the Riemann tensor. We start with the definition we need.

[2.1] *Definition.* Let R be a rank (3, 1) tensor and let R be antisymmetric as in [Definition 5.1]. We call R *broad* at m if for every vector v at m there are two vectors w and x at m such that $R(w,x)v$ does not lie in the plane generated by w and x, that is

$$\{R(w,x)v, w, x\}$$

is linearly independent.

This condition does not have any clear geometric content; but seems to be a fairly reasonable assumption to make on a space-time.

We are now ready to prove our theorem.

[2.2] *Theorem.* Let *Dim(M)* $\geqslant 4$. Let R be the Riemann tensor of a pseudo-metric. Suppose R is broad and total [see 5.1] at every point in M. Then this pseudo-metric is the unique pseudo-metric (to within a constant conformal factor) that has R as its curvature. Also any constant times this pseudometric has R as its curvature.

Proof. The last statement is easy to see since if $\nabla g = 0$ then $\nabla(kg) = k\nabla g = 0$, and g has the same connection as kg.

We now suppose we have two pseudo-metrics g and $\bar{g}$ with the same curvature R. Since R is total we know from [5.2] that

$$\bar{g} = e^{2\alpha} g$$

where α is a function from M to $\mathbb{R}$. We must only show α is a constant. To do this we shall use the Bianchi identities (see Section 5 for notation):

$$R^l_{i,jk\|n} + R^l_{i,nj\|k} + R^l_{i,kn\|j} = 0 \tag{2.1}$$

$$R^l_{i,jk} + R^l_{j,ki} + R^l_{k,ij} = 0 \tag{2.2}$$

The first equation involves the covariant derivative of the given metric. We use $\Gamma^k_{ij}(\bar{\Gamma}^k_{ij})$ to indicate the connection derived from $g_{ij}(\bar{g}_{ij})$. Rewriting (2.1) we find

$$\begin{aligned} 0 = R^l_{i,jk|n} + R^l_{i,nj|k} + R^l_{i,kn|j} + C^l_{pn}R^p_{i,jk} - C^p_{in}R^l_{p,jk} \\ + C^l_{pk}R^p_{i,nj} - C^p_{ik}R^l_{p,nj} + C^l_{pj}R^p_{i,kn} - C^p_{ij}R^l_{p,kn} \end{aligned} \tag{2.3}$$

where C^i_{jk} may be taken to be either Γ^i_{jk} or $\bar{\Gamma}^i_{jk}$. Now we observe that

$$\bar{\Gamma}^i_{jk} - \Gamma^i_{jk} = \alpha_{|j}\delta_k{}^i + \alpha_{|k}\delta_j{}^i - \alpha_{|n}g^{ni}g_{jk} \tag{2.4}$$

Let $\bar{\Gamma}^i_{jk} - \Gamma^i_{jk} = D^i_{jk}$. Then subtracting the previous equation when $C = \Gamma$ from the same equation when $C = \bar{\Gamma}$ we find

$$\begin{aligned} 0 = D^l_{pn}R^p_{i,jk} - D^p_{in}R^l_{p,jk} + D^l_{pk}R^p_{i,nj} - D^p_{ik}R^l_{p,nj} \\ + D^l_{pj}R^p_{i,kn} - D^p_{ij}R^l_{p,kn} \end{aligned} \tag{2.5}$$

Using our expression for D in terms of α and g_{ij} we find

$$\begin{aligned} 0 = {} & \delta_m{}^l (\alpha_{|n} R^{ni}{}_{,jk}) + \delta_k{}^l (\alpha_{|n} R^{ni}{}_{,mj}) + \delta_j{}^l (\alpha_{|n} R^{ni}{}_{,km}) \\ & + \delta_m{}^i (\alpha_{|p} R^l_{n,jk} g^{pn}) + \delta_k{}^i (\alpha_{|p} R^l_{n,mj} g^{np}) + \delta_j{}^i (\alpha_{|p} R^l_{n,km} g^{mp}) \end{aligned} \tag{2.6}$$

This equation will hold in any coordinate system. We will now pick a particular coordinate system. Suppose α is not constant. Then there is a point m_0 in M such that $d\alpha \neq 0$ at m_0. Thus $\alpha_{|p} g^{pn}$ will be a nonzero vector at m_0. We now use [2.1] by taking $v = \alpha_{|p} g^{pn}$. Since R is broad there is a w and x such that

$$\{R(w, x)v, w, x\} \tag{2.7}$$

is independent. Thus there is a coordinate system in which

$$\frac{\partial}{\partial x^1} = w, \qquad \frac{\partial}{\partial x^2} = x, \qquad \frac{\partial}{\partial x^3} = R(w, x)v \text{(see footnote 1)} \tag{2.8}$$

Now in this coordinate system we will use the previous equation with

$$j = 1 \quad k = 2 \quad l = 3 \quad m = i = 4 \tag{2.9}$$

We find

$$0 = \alpha_{|p} g^{pn} R^3_{n,21} \tag{2.10}$$

but

$$\frac{\partial}{\partial x^3} = R\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right) v = \alpha_{|p} g^{pn} R^q_{n,21} \frac{\partial}{\partial x^q} \tag{2.11}$$

which implies $\alpha_{|p} g^{pn} R^3_{n,21} = 1 \neq 0$. This gives the desired contradiction.

[2.3] *Corollary*. Let R be as in [2.2]. There is a unique pseudo-metric connection which has R as its curvature.

In order to illustrate the necessity of another condition than totality in [2.2] we give the following example:

[2.4] *Example*. Let $ds_1{}^2 = e^{2x^1} (dx^{0^2} - dx^{1^2})$ and $ds_2{}^2 = 1/x^0 \; e^{2x^1} (dx^{0^2} - dx^{1^2})$. Then $ds_1{}^2$ and $ds_2{}^2$ have the same Riemann curvature and this curvature tensor is total ($R^l_{0,10} = 2$).

Having established the uniqueness theorem we may state an existence theorem which is an easy consequence of this uniqueness theorem. First, a technical lemma will be needed for the sake of mathematical completeness. We need to know that if R^l_{ijk} are smooth functions then $G(R^l_{ijk})$ (see [5.3]) will be smooth. This is not obvious from the formula for G since [5.2] (b) is not smooth.

[1] Given a set of j independent vectors v_i at a point m_0 expand this basis v_j of $T_{m_0}(M)$. Define the local chart ϕ: $\mathbb{R}^n \to M$ by $\phi(\lambda^j) = \exp_{m_0}(\Sigma \lambda^i v_j)$. In this chart $v_i = \partial/\partial x_i$.

[2.5] *Lemma.* If $R^l_{i,jk}$ are smooth functions then so are the $G(R^l_{i,jk})$.

Proof. The distribution D generated by v_α (defined in [5.2](e)) is smooth if the $R^l_{i,jk}$ are smooth. This is a distribution taking a point in space-time into an a-1 dimensional subspace of an a dimensional normed vector space ([5.2](f)). Since D is smooth then $D^\perp = \{v/(v, D) = 0\}$ will be smooth, and so $D^\perp \cap \{v/(v, v) = 1\}$ will be also smooth. Now (see [5.2])

$$D^\perp \cap \{v/(v, v) = 1\} = \{\pm w/(w, w)^{1/2}\} \tag{2.12}$$

so that $w/(w, w)^{1/2}$ will be a smooth function of $R^l_{i,jk}$ in any simply connected region of M. Since smoothness is a local concept $w/(w, w)^{1/2}$ will be smooth. Thus the projections

$$G_{ij}(R) = (w/(w, w)^{1/2}, x_{ij}) \tag{2.13}$$

will be smooth functions.

Now for some notation.

[2.6] *Definition.* Let R be a rank (3, 1) tensor and let R be antisymmetric as in [5.1]. Suppose $G(R)$ is a pseudo-metric. Let α be a real valued function on the manifold. Define

(1) $P(R^l_{i,jk}) = R^l_{i,jk|n} + R^l_{i,nj|k} + R^l_{i,kn|j}$

(2) $F(\alpha, R) = -C^l_{pn}R^p_{i,jk} + C^p_{in}R^l_{p,jk} - C^l_{pk}R^p_{i,nj} + C^p_{ik}R^l_{p,nj} - C^l_{pj}R^p_{i,kn} + {}$
$+ C^p_{ij}R^l_{p,kn}$

where $C^i_{jk} = \Gamma^i_{jk} + \alpha_{|j}\delta_k{}^i + \alpha_{|k}\delta_j{}^i - \alpha_{|n}g^{ni}g_{jk}$ and Γ^i_{jk} is the connection associated with $G(R)$ considered as a pseudo-metric.

Notice $F(\alpha, R) = P(R)$ is a linear equation in α. This equation is in fact the bianchi equations for the conformal factor $e^{2\alpha}$ that must multiply $G(R)$ in order to obtain a pseudo-metric with curvature R. Now we present our theorem.

[2.7] *Theorem.* Let R be a rank (3, 1) tensor and let R be antisymmetric as in [5.1]. Suppose R is both total and broad ([5.1] and [2.1]) and $\dim M \geqslant 4$. Then R is the curvature tensor of a pseudo-metric of a given signature s if and only if

(a) $G(R)$ has signature s.

(b) The linear equation (see [2.6])

$$F(\alpha, R) = P(R)$$

has a solution for α.

(c) Riem $(e^{2\alpha}G(R)) = R$
where Riem (pseudo-metric) is the Riemann curvature of that pseudo-metric.

Proof. Suppose R is the curvature of a pseudo-metric g. (a) must hold since $G(R)$ is conformally related to g of the proper signature [5.2]. As for (b) we find in the proof of [2.2] that $F(\alpha, R) = P(R)$ is just the bianchi identity of g if

$$g = e^{2\alpha} G(R)$$

Again such an α exists by [5.2]. Now (c) is only the statement that g has R for its curvature as we assumed.

Now let us assume (a), (b), (c) hold. First of all $G(R)$ will be a smooth symmetric tensor by [2.5] (symmetry follows from the definition of $G(R)$). (a) assures us that $G(R)$ will be a pseudo-metric. (c) says that R is the curvature of $e^{2\alpha}G(R)$ which says that $e^{2\alpha}G(R)$ is the desired pseudo-metric.

[2.7] reduces the existence problem of finding a g with a given curvature to finding the solution to a given system of linear equations. If such a solution does not exist, the g cannot exist. If a solution does exist then g will exist as long as the consistency equation [2.7] (c) is satisfied. This equation involves only the R^1_{ijk} once α is known. Thus the problem of existence is completely solved once one can solve [2.7] (b).

3. *The Conformal Factor in Terms of the Holonomy Group*

In this Section we will show that information about the holonomy group will completely determine the conformal factor. The proof will be more geometric in nature and will serve to explain why the proof of [2.2] involved only the connection and not the metric. The holonomy group and the Riemann Curvature are very closely related so that the results of this Section are connected to those of Section 2. However the techniques are apparently quite different. Thus this Section presents a different side to the uniqueness problem.

We will start the Section by establishing some notation.

[3.1] *Notation.* Let g be a pseudo-metric on M and γ a path in M from $\gamma(0)$ to $\gamma(1)$.
(a) Denote by ${}_\gamma T$ the map

$${}_\gamma T: T_{\gamma(0)}(M) \to T_{\gamma(1)}(M)$$

where ${}_\gamma T(v)$ is the parallel transport of v along γ.
(b) Let $\mathcal{L}_m = \{\gamma/\gamma$ is a closed path from m to m (i.e. $\gamma(0) = \gamma(1) = m$) and γ is null homotopic$\}$
where $m \in M$.
(c) Let ϕ_m denote the identity component of the holonomy group at m, i.e.

$$\phi_m = \{{}_\gamma T/\gamma \in \mathcal{L}_m\}$$

(d) Denote by H the map

$$H: \mathcal{L}_m \to \phi_m$$

$$H(\gamma) = {}_\gamma T$$

Now that we have the necessary notation for parallel transport and holonomy we just need some notation for how to combine paths together.

[3.2] *Notation.* Let γ_i be a path from $\gamma_i(0)$ to $\gamma_i(1)$ for $i = 1, 2$.
(a) If $\gamma_1(1) = \gamma_2(0)$ then define $\gamma_2{*}\gamma_1$ to be the path created by first following γ_1 and then γ_2, i.e.

$$\gamma_2{*}\gamma_1(t) = \begin{cases} \gamma_1(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma_2(2t-1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Note

$$\gamma_2{*}\gamma_1(0) = \gamma_1(0), \qquad \gamma_2{*}\gamma_1(1) = \gamma_2(1)$$

(b) Define $\bar{\gamma}_1$ to be the path obtained by going backwards along γ_1 from $\gamma_1(1)$ to $\gamma_1(0)$, i.e.

$$\bar{\gamma}_1(t) = \gamma_1(1-t)$$

Some of the elementary properties of parallel transport are as follows.

[3.3] *Observation.* (a) ${}_{\gamma_1 * \gamma_2}T = {}_{\gamma_1}T_{\gamma_2}T$
(b) ${}_{\bar{\gamma}}T = ({}_{\gamma}T)^{-1}$

Before giving our theorem we state a well known lemma from group theory. For the sake of completeness we present a short proof of this lemma.

[3.4] *Lemma.* Let $0(i, j)$ be a group that preserves a nondegenerate $i + j$ dimensional form with signature $i - j$. Let A be any transformation such that

$$ABA^{-1} = B \text{ for all } B \in 0(i, j).$$

Then $A = aI$ where a is some constant and I is the identity transformation.

Proof. Extend all transformations which act on $\mathbb{R}^n$ to transformations that act on $\mathbb{C}^n$ in the natural manner. A must have a nontrivial eigenspace V corresponding to eigenvalue a. Now $0(i, j)$: $V \to V$ since if $v \in V$ and $B \in 0(i, j)$ we have

$$A(B(v)) = B(A(v)) = aB(v)$$

and $B(v) \in V$. But the only two subspaces of $\mathbb{C}^n$ fixed by $0(i, j)$ are $\mathbb{C}^n$ and 0. Thus $\mathbb{C}^n = V$ which says $A = aI$. a is real since A was real.

We are now ready to present our theorem. If we have two pseudo-metrics g and $\bar{g}$ with the same curvature and if that curvature is total then the two holonomy groups ϕ and $\bar{\phi}$ will be the same. However the map H and $\bar{H}$ may not be (see [3.1] (d)) as the example [2.3] illustrates. If H does equal $\bar{H}$ though, then g must equal $\bar{g}$. We show this in the following theorem.

[3.5] *Theorem.* Let $\dim M > 1$ and let g and $\bar{g}$ be two pseudo-metrics with the same curvature R which is total. If $H = \bar{H}$ (see [3.1] (d)) then $g = k\bar{g}$ where k is some constant.

Proof. We start by trying to find a relationship between the two different transport laws ${}_\gamma T$ and ${}_\gamma \bar{T}$ corresponding to g and $\bar{g}$. Let m_0 be a fixed point and γ any path going to a point m. Now if

$$\alpha \in \mathscr{L}_{m_0} \quad \text{then} \quad \gamma * \alpha * \bar{\gamma} \in \mathscr{L}_m$$

We have $H(\alpha) = \bar{H}(\alpha)$ and $H(\gamma * \alpha * \bar{\gamma}) = \bar{H}(\gamma * \alpha * \bar{\gamma})$
Thus

$${}_\alpha T = {}_\alpha \bar{T}$$

and

$${}_\gamma T_\alpha T_\gamma T^{-1} = {}_\gamma \bar{T}_\alpha \bar{T}_\gamma \bar{T}^{-1}$$

using [3.1](d) and [3.3]. Combining we find

$$({}_\gamma \bar{T}^{-1}{}_\gamma T)_\alpha T({}_\gamma \bar{T}^{-1}{}_\gamma T)^{-1} = {}_\alpha T$$

Now $\{{}_\alpha T / \alpha \in \mathscr{L}_{m_0}\} = H_{m_0}$ which is all of $O(j, k)$ since R is total. (R is total means the $O(j, k) \simeq W$ in [5.1] and W is determined completely in terms of R). Thus we may apply [3.4] to the above relation to find that

$${}_\gamma T = a(\gamma)_\gamma \bar{T} \tag{3.1}$$

where a is some constant that depends on γ. We now show that a depends only on the end point of γ which is m. Suppose γ_1 and γ_2 go from m_0 to m. Then

$$\bar{\gamma}_1 * \gamma_2 \in \mathscr{L}_{m_0} \tag{3.2}$$

so

$$({}_{\gamma_1}T)^{-1}{}_{\gamma_2}T = ({}_{\gamma_1}\bar{T})^{-1}{}_{\gamma_2}\bar{T} \tag{3.3}$$

This leads to

$$\frac{a(\gamma_2)}{a(\gamma_1)}{}_{\gamma_1}\bar{T}^{-1}{}_{\gamma_2}\bar{T} = {}_{\gamma_1}\bar{T}^{-1}{}_{\gamma_2}\bar{T} \tag{3.4}$$

which gives $a(\gamma_2)/a(\gamma_1) = 1$ as was desired. Clearly a is a smooth function so we find that the parallel transports are 'conformally related.' The rest of the proof just establishes the fact that conformally related metrics cannot have conformally related transport laws. Observe that g and $\bar{g}$ must be conformally related because of [5.2]. One can see this directly without the use of [5.2] by the following argument: Since $\phi_{m_0} = O(j, k)$, $\bar{g}_{m_0} = \lambda g_{m_0}$ where λ is some constant. Now let γ be any path from m_0 to m and v, w be two vectors at m.

$$\begin{aligned} \bar{g}_m(v, w) &= \bar{g}_{m_0}({}_{\bar{\gamma}}\bar{T}(v), {}_{\bar{\gamma}}\bar{T}(w)) = \lambda g_{m_0}(a(m)_{\bar{\gamma}}T(v), a(m)_{\bar{\gamma}}T(w)) \\ &= \lambda a(m)^2 g_{m_0}({}_{\bar{\gamma}}T(v), {}_{\bar{\gamma}}T(w)) \\ &= \lambda a(m)^2 g_m(v, w) \end{aligned} \tag{3.5}$$

so $\bar{g} = \lambda a^2 g$.

Now we assume $\bar{g}_{ij} = e^{2\alpha} g_{ij}$ so that

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \alpha_{|j}\delta_k{}^i + \alpha_{|k}\,\delta_j{}^i - \alpha_{|n} g^{ni} g_{jk} \tag{3.6}$$

We also assume ${}_\gamma\bar{T} = \beta T_\gamma$ so that if v^i is a parallel vector field along γ then βv^i will be parallel along γ in $\bar{\Gamma}$. The two equations expressing the parallelism are

$$\dot{\gamma}^i v^k_{|i} + \Gamma^k_{ij}\dot{\gamma}^i v^j = 0 \tag{3.7}$$

and

$$\frac{1}{\beta}\,\dot{\gamma}^i(\beta v^k)_{|i} + \bar{\Gamma}^k_{ij}\dot{\gamma}^i v^j = 0 \tag{3.8}$$

Subtracting these two equations we find

$$(\dot{\gamma}^i(\ln\beta)_{|i})v^k + (\alpha_{|i}\dot{\gamma}^i)v^k + (\alpha_{|i}v^i)\dot{\gamma}^k + \alpha_{|n} g^{nk} v^i \dot{\gamma}^j g_{ij} = 0 \tag{3.9}$$

Using this equation we will show $\alpha_{|i} = 0$ at every point m in M. Thus α will be constant as desired. Suppose $\alpha_{|i} \neq 0$ at some point m. Then $\alpha_{|j} g^{ij}$ will be a nonzero vector. Thus there is a non-null vector v^i at m such that

$$0 \neq \alpha_{|j} g^{ji} v^k g_{ik} = \alpha_{|i} v^k \tag{3.10}$$

There is another non-null vector w^k such that

$$w^k v^j g_{jk} = 0 \tag{3.11}$$

since v^j is non-null. Now let γ be any curve with tangent w^j at m. Let v^j be defined on this curve as the parallel transport of v^j at m along the curve. Thus the above equation holds for $w^j = \dot{\gamma}^j$ and

$$\begin{aligned} 0 &= (\dot{\gamma}^k(\ln\beta)_{|i} + \alpha_{|i}\dot{\gamma}^i)v^k w^1 g_{kl} + (\alpha_{|n}w^n)v^i w^j g_{ij} \\ &= (\alpha_{|i}v^i)w^k w^1 g_{kl} \\ &= (\alpha_{|i}v^i)w^k w^1 g_{kl} \end{aligned} \tag{3.12}$$

but $w^k w^l g_{kl} \neq 0$ since w^k was non-null and so $\alpha_{|i}v^i = 0$ giving our needed contradiction.

The basic idea of this proof is that H partially determines the metric and partially determines the connection. Then it is seen that only one metric will satisfy both of the limitations. Thus, when working with R^l_{ijk} as in Section 2 one would first look for a way to determine Γ^k_{ij}. The only identity involving the Riemann tensor and the connection is the Bianchi identity involving covariant derivatives. Thus it would seem reasonable that one should work with this identity to get a complete uniqueness theorem, and in fact this approach did work although it seems necessary to have the extra restriction on R^l_{ijk} given in [2.2].

4. *Discussion*

In this Section we would like to discuss the significance the preceding work has in general relativity from a philosophical point of view. A fundamental problem in the general theory since it first was proposed has been that it does not seem to be compatible with Mach's principle. There are many aspects to Mach's principle. Let us first consider the particular aspect that is relevant to the results we have obtained here; we will then consider Mach's principle from a broader point of view.

Let us suppose that we are given a test particle in a completely empty universe with no gravitation field and asked how this particle will move. One might at first say the particle will move in a straight line as would be demanded in the Minkowskian model. However a "rotating" metric

$$ds^2 = -dt^2 + dR^2 + R^2(d\theta + \theta_0\, dt)^2 + R^2(\sin(\theta + t\theta_0))^2\, d\phi^2 \tag{4.1}$$

also has no gravitational field. This metric will have Coreolus forces and particles will not travel in straight lines. How does one choose between these two different situations? Another way of looking at the problem is as follows: In order to describe the motion of the particle we must first establish a coordinate system. Once having established a system how do we determine if we have picked an "inertial" system, a "rotating" system, or some other entirely different system? Since we have no physical information to draw upon, there can be no way of resolving the problem. This poses a dilemma which can be formulated in the following mathematical way. There are gravitational fields (in this case $R^l_{ijk} = 0$) which are compatible with more than one force law for the motion of particles (i.e., more than one connection). How does one choose the force law when all that one really has at one's disposal are the gravitational fields?

Einstein dealt with this problem by making the following observation. The problem with our example is that a completely empty space is a physically impossible situation. In any realistic situation one must have matter in the universe and using this matter one can decide which frames are inertial and which are not. This idea translated into mathematical language would say that if one restricts oneself to gravitational fields that are "sufficiently non-zero" then there will in fact be a unique force law that corresponds to that field. This is nothing more than a part of Mach's principle which says that the physics of the universe is determined by the matter in the universe (and thus by the gravitational fields also since they determine the matter distribution of the universe via Einstein's equations). [2.3] is a theorem which in fact gives conditions on R which assure that Γ will be unique. These conditions are very general and it seems they will be satisfied as long as the space-time is not allowed to be too empty. Thus this theorem says that by restricting ourselves to a class of realistic models this aspect of Mach's principle will be satisfied.

Let us now make a precise list of several aspects of Mach's principle and review their status. The first principle is the one that is the general principle we wish to consider. All the other principles can be considered as steps towards this principle. This principle says that the mass distribution of a universe completely determines the physics of the universe:

[4.1] *Definition.* Mach's principle 1 (Mach 1) applies to a class of space-times $\mathscr{C}$ if for any two space-times in $\mathscr{C}$ that have the same energy tensor T_{ij} will also have the same connection Γ^k_{ij}.

The principles that build up to this one may be listed as follows:

[4.2] *Definition.* Mach's principle 2 (Mach 2) applies to $\mathscr{C}$ if any two space-times in $\mathscr{C}$ with the same R^l_{ijk} have the same Γ^k_{ij}.

[4.3] *Definition.* Mach's principle 3 (Mach 3) applies to $\mathscr{C}$ if any two space-times in $\mathscr{C}$ with the same T_{ij} have the same R^l_{ijk}.

[4.4] *Definition.* Mach's principle 4 (Mach 4) applies to $\mathscr{C}$ if any space-time in $\mathscr{C}$ with $R_{ij} = 0$ satisfies $R^l_{ijk} = 0$.

In order to make Mach's principle more compatible with general relativity one must look for a class $\mathscr{C}$ of space-time that has Mach 1 and contains enough space-times to make reasonable physical models.

Here we have given such a theorem for Mach 2. The restriction on spacetimes in this case seems to be so mild that this theorem will probably be adequate for Mach 2. In Ihrig, (1975) we gave a theorem for Mach 4 (see Ozsvath, (1962) for examples of spacetimes that do not satisfy Mach 4).

In this theorem $\mathscr{C}$ is taken to be a subclass of the periodic spacetimes (see Ihrig and Sen, 1974) which is a very restricted class (although sufficiently large to satisfy some physically desirable properties). There is work still left to be done to try to expand the $\mathscr{C}$ for Mach 4.

However the most important unsolved (and largely unconsidered) problem is Mach 3. One would hope that the class $\mathscr{C}$ for Mach 4 would be sufficiently restrictive for Mach 3, but this problem remains unsolved.

5. *Appendix*

Here we would like to recall some notation and results (Ihrig, 1975a). First we establish notation for the Riemann tensor. We take

$$R(x,y)z = D_x D_y z - D_y D_x z - D_{[x,y]} z \tag{5.1}$$

In (Ihrig, 1975a) we defined $R^l_{ij,k}$ by

$$R^l_{ij,k} \frac{\partial}{\partial x^l} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k} \tag{5.2}$$

since $R^l_{ij,k}$ was only considered as a matrix with indices l and k (i and j being fixed). Since the $R^l_{ij,k}$ is not used here in this way we will return to the standard notation

$$R^l_{i,jk}\frac{\partial}{\partial x^l}=R\left(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^i} \tag{5.3}$$

We also use

$$F_{|i}=\frac{\partial}{\partial x^i}F \tag{5.4}$$

where F is any function. If T is a tensor then

$$T_{\|i}=D_{\partial/\partial x^i}T \tag{5.5}$$

Now we will state the definition and theorem we use:

[5.1] *Definition.* Let R be a tensor of rank (3, 1) and let R be antisymmetric in two indices:

$$R(v,w)=-R(w,v)$$

Then R is called total at m if

$$\{R_m(v,w)\,|\,v,w\in T_m(M)\}=W$$

generates a vector space of dimension $n(n-1)/2$ where $n=\dim M$.

[5.2] *Theorem.* Let R be the Riemann tensor of some pseudo-metric. If R is total at m then g_{ij} is determined to within a conformal factor at m by the following equations (x_{ij} are independent variables).

(a) $\lambda g_{ij}=(w,x_{ij})/(w,w)^{1/2}$

(b) $w=\sum\limits_{\alpha=a+1}^{b}w_\alpha,\qquad \begin{cases}a=(n-1)n^{3/2}\\ b=a+n(n+1)/2\end{cases}$

(c) $w_\alpha=\overline{w}_\alpha/(\overline{w}_\alpha,\overline{w}_\alpha)^{1/2}\qquad \text{if } w_\alpha\neq 0$

$\quad\; w_\alpha=0\qquad\qquad\qquad\qquad \text{if } w_\alpha=0$

(d) $\overline{w}_\alpha=v_\alpha-\sum\limits_{\beta<\alpha}(v_\alpha,w_\beta)w_\beta$

(e) $v_\alpha=\begin{cases}x_{11'}R^{1'}_{k,ij}+x_{k1'}R^{1'}_{1,ij} & \alpha<a\\ x_{ij} & \alpha\geqslant a\end{cases}$

(f) $(x_{i_1j_1},x_{i_2j_2})=\delta_{i_1i_2}\delta_{j_1j_2}$

[5.3] *Notation.* Let R be as in the above theorem. Let

$$\lambda g_{ij} = G(R)$$

which is determined by the above prescription.

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